UFRGS – INSTITUTO DE MATEMÁTICA
Departamento de Matemática Pura e Aplicada
MAT01168 - Turma A - 2019/2
Prova da área IIB

1 - 5	6	7	Total

Nome: ______________________________ **Cartão:** ______________

Regras Gerais:

- Não é permitido o uso de calculadoras, telefones ou qualquer outro recurso computacional ou de comunicação.
- Trabalhe individualmente e sem uso de material de consulta além do fornecido.
- Devolva o caderno de questões preenchido ao final da prova.

Regras para as questões abertas:

- Seja sucinto, completo e claro.
- Justifique todo procedimento usado.
- Indique identidades matemáticas usadas, em especial, itens da tabela.
- Use notação matemática consistente.

Propriedades das transformadas de Fourier: considere a notação $F(w) = \mathcal{F}\{f(t)\}$.

1.	Linearidade	$\mathcal{F}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{F}\{f(t)\} + \beta\mathcal{F}\{g(t)\}$
2.	Transformada da derivada	Se $\lim_{t\to\pm\infty} f(t) = 0$, então $\mathcal{F}\{f'(t)\} = iw\mathcal{F}\{f(t)\}$ Se $\lim_{t\to\pm\infty} f(t) = \lim_{t\to\pm\infty} f'(t) = 0$, então $\mathcal{F}\{f''(t)\} = -w^2\mathcal{F}\{f(t)\}$
3.	Deslocamento no eixo w	$\mathcal{F}\{e^{at}f(t)\} = F(w + ia)$
4.	Deslocamento no eixo t	$\mathcal{F}\{f(t-a)\} = e^{-iaw}F(w)$
5.	Transformada da integral	Se $F(0) = 0$, então $\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{F(w)}{iw}$
6.	Teorema da modulação	$\mathcal{F}\{f(t)\cos(w_0t)\} = \frac{1}{2}F(w-w_0) + \frac{1}{2}F(w+w_0)$
7.	Teorema da Convolução	$\mathcal{F}\{(f*g)(t)\} = F(w)G(w)$, onde $(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ $(F*G)(w) = 2\pi\mathcal{F}\{f(t)g(t)\}$
8.	Conjugação	$\overline{F(w)} = F(-w)$
9.	Inversão temporal	$\mathcal{F}\{f(-t)\} = F(-w)$
10.	Simetria ou dualidade	$f(-w) = \frac{1}{2\pi}\mathcal{F}\{F(t)\}$
11.	Mudança de escala	$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F\left(\frac{w}{a}\right)$, $a \neq 0$
12.	Teorema da Parseval	$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(w)\rvert^2 dw$
13.	Teorema da Parseval para Série de Fourier	$\frac{1}{T}\int_0^T \lvert f(t)\rvert^2 dt = \sum_{n=-\infty}^{\infty} \lvert C_n\rvert^2$

Séries e transformadas de Fourier:

	Forma trigonométrica	Forma exponencial
Série de Fourier	$f(t) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n\cos(w_nt) + b_n\,\text{sen}(w_nt)]$ onde $w_n = \frac{2\pi n}{T}$, T é o período de $f(t)$ $a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$, $a_n = \frac{2}{T}\int_0^T f(t)\cos(w_nt)dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(w_nt)dt$, $b_n = \frac{2}{T}\int_0^T f(t)\,\text{sen}(w_nt)dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\text{sen}(w_nt)dt$	$f(t) = \sum_{n=-\infty}^{\infty} C_ne^{iw_nt}$, onde $C_n = \frac{a_n - ib_n}{2}$
Transformada de Fourier	$f(t) = \frac{1}{\pi}\int_0^{\infty}(A(w)\cos(wt) + B(w)\,\text{sen}(wt))\,dw$, para $f(t)$ real, onde $A(w) = \int_{-\infty}^{\infty} f(t)\cos(wt)dt$ e $B(w) = \int_{-\infty}^{\infty} f(t)\,\text{sen}(wt)dt$	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dw$, onde $F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt$

Tabela de integrais definidas:

1. $\displaystyle\int_0^\infty e^{-ax}\cos(mx)dx = \frac{a}{a^2+m^2} \qquad (a>0)$	2. $\displaystyle\int_0^\infty e^{-ax}\,\mathrm{sen}(mx)dx = \frac{m}{a^2+m^2} \qquad (a>0)$
3. $\displaystyle\int_0^\infty \frac{\cos(mx)}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ma} \qquad (a>0,\ m\geq 0)$	4. $\displaystyle\int_0^\infty \frac{x\,\mathrm{sen}(mx)}{a^2+x^2}dx = \frac{\pi}{2}e^{-ma} \qquad (a\geq 0,\ m>0)$
5. $\displaystyle\int_0^\infty \frac{\mathrm{sen}(mx)\cos(nx)}{x}dx = \begin{cases} \frac{\pi}{2}, & n<m \\ \frac{\pi}{4}, & n=m, \quad (m>0,\ n>0) \\ 0, & n>m \end{cases}$	6. $\displaystyle\int_0^\infty \frac{\mathrm{sen}(mx)}{x}dx = \begin{cases} \frac{\pi}{2}, & m>0 \\ 0, & m=0 \\ -\frac{\pi}{2}, & m<0 \end{cases}$
7. $\displaystyle\int_0^\infty e^{-r^2x^2}dx = \frac{\sqrt{\pi}}{2r} \qquad (r>0)$	8. $\displaystyle\int_0^\infty e^{-a^2x^2}\cos(mx)dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{m^2}{4a^2}} \qquad (a>0)$
9. $\displaystyle\int_0^\infty xe^{-ax}\,\mathrm{sen}(mx)dx = \frac{2am}{(a^2+m^2)^2} \qquad (a>0)$	10. $\displaystyle\int_0^\infty e^{-ax}\,\mathrm{sen}(mx)\cos(nx)dx = \frac{m(a^2+m^2-n^2)}{(a^2+(m-n)^2)(a^2+(m+n)^2)} \qquad (a>0)$
11. $\displaystyle\int_0^\infty xe^{-ax}\cos(mx)dx = \frac{a^2-m^2}{(a^2+m^2)^2} \qquad (a>0)$	12. $\displaystyle\int_0^\infty \frac{\cos(mx)}{x^4+4a^4}dx = \frac{\pi}{8a^3}e^{-ma}(\mathrm{sen}(ma)+\cos(ma))$
13. $\displaystyle\int_0^\infty \frac{\mathrm{sen}^2(mx)}{x^2}dx = \lvert m\rvert\frac{\pi}{2}$	14. $\displaystyle erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$
15. $\displaystyle\int_0^\infty \frac{\mathrm{sen}^2(ax)\,\mathrm{sen}(mx)}{x}dx = \begin{cases} \frac{\pi}{4}, & (0<m<2a) \\ \frac{\pi}{8}, & (0<2a=m) \\ 0, & (0<2a<m) \end{cases}$	16. $\displaystyle\int_0^\infty \frac{\mathrm{sen}(mx)\,\mathrm{sen}(nx)}{x^2}dx = \begin{cases} \frac{\pi m}{2}, & (0<m\leq n) \\ \frac{\pi n}{2}, & (0<n\leq m) \end{cases}$
17. $\displaystyle\int_0^\infty x^2e^{-ax}\,\mathrm{sen}(mx)dx = \frac{2m(3a^2-m^2)}{(a^2+m^2)^3} \qquad (a>0)$	18. $\displaystyle\int_0^\infty x^2e^{-ax}\cos(mx)dx = \frac{2a(a^2-3m^2)}{(a^2+m^2)^3} \qquad (a>0)$
19. $\displaystyle\int_0^\infty \frac{\cos(mx)}{(a^2+x^2)^2}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma} \qquad (a>0,\ m\geq 0)$	20. $\displaystyle\int_0^\infty \frac{x\,\mathrm{sen}(mx)}{(a^2+x^2)^2}dx = \frac{\pi m}{4a}e^{-ma} \qquad (a>0,\ m>0)$
21. $\displaystyle\int_0^\infty \frac{x^2\cos(mx)}{(a^2+x^2)^2}dx = \frac{\pi}{4a}(1-ma)e^{-ma} \qquad (a>0,\ m\geq 0)$	22. $\displaystyle\int_0^\infty xe^{-a^2x^2}\,\mathrm{sen}(mx)dx = \frac{m\sqrt{\pi}}{4a^3}e^{-\frac{m^2}{4a^2}} \qquad (a>0)$

Identidades Trigonométricas:

$\cos(x)\cos(y) = \dfrac{\cos(x+y)+\cos(x-y)}{2}$
$\mathrm{sen}(x)\,\mathrm{sen}(y) = \dfrac{\cos(x-y)-\cos(x+y)}{2}$
$\mathrm{sen}(x)\cos(y) = \dfrac{\mathrm{sen}(x+y)+\mathrm{sen}(x-y)}{2}$

Frequências das notas musicais em hertz:

Nota \ Escala	2	3	4	5	6	7
Dó	65,41	130,8	261,6	523,3	1047	2093
Dó ♯	69,30	138,6	277,2	554,4	1109	2217
Ré	73,42	146,8	293,7	587,3	1175	2349
Ré ♯	77,78	155,6	311,1	622,3	1245	2489
Mi	82,41	164,8	329,6	659,3	1319	2637
Fá	87,31	174,6	349,2	698,5	1397	2794
Fá ♯	92,50	185,0	370,0	740,0	1480	2960
Sol	98,00	196,0	392,0	784,0	1568	3136
Sol ♯	103,8	207,7	415,3	830,6	1661	3322
Lá	110,0	220,0	440,0	880,0	1760	3520
Lá ♯	116,5	233,1	466,2	932,3	1865	3729
Si	123,5	246,9	493,9	987,8	1976	3951

Integrais:

$\displaystyle\int xe^{\lambda x}\,\mathrm{d}x = \frac{e^{\lambda x}}{\lambda^2}(\lambda x-1)+C$
$\displaystyle\int x^2e^{\lambda x}\,\mathrm{d}x = e^{\lambda x}\left(\frac{x^2}{\lambda}-\frac{2x}{\lambda^2}+\frac{2}{\lambda^3}\right)+C$
$\displaystyle\int x^ne^{\lambda x}\,\mathrm{d}x = \frac{1}{\lambda}x^ne^{\lambda x}-\frac{n}{\lambda}\int x^{n-1}e^{\lambda x}\mathrm{d}x+C$
$\displaystyle\int x\cos(\lambda x)\,dx = \frac{\cos(\lambda x)+\lambda x\,\mathrm{sen}(\lambda x)}{\lambda^2}+C$
$\displaystyle\int x\,\mathrm{sen}(\lambda x)\,dx = \frac{\mathrm{sen}(\lambda x)-\lambda x\cos(\lambda x)}{\lambda^2}+C$

• **Questão 1** (1.0 ponto) Resolva o seguinte problema de difusão de calor:

$$u_t(x,t) - 2u_{xx}(x,t) = 0$$
$$u(x,0) = \delta(x).$$

Assinale na primeira coluna a transformada de Fourier $U(k,t) = \mathcal{F}\{u(x,t)\}$ e na segunda a solução $u(x,t)$.

( ) $U(k,t) = \frac{1}{\sqrt{\pi t}} e^{-2k^2 t}$

( ) $U(k,t) = e^{-k^2 t}$

( ) $U(k,t) = e^{-ik} e^{-2k^2 t}$

( ) $U(k,t) = e^{-2k^2 t}$

( ) $U(k,t) = \frac{e^{-ik}}{\sqrt{\pi t}} e^{-2k^2 t}$

( ) $u(x,t) = \frac{1}{2\sqrt{2\pi t}} e^{-\frac{x^2}{8t}}$

( ) $u(x,t) = e^{-\frac{x^2}{8t}}$

( ) $u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4t}}$

( ) $u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-1)^2}{8t}}$

( ) $u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x+1)^2}{4t}}$

• **Questão 2** (1.0 ponto) Considere a função $f(t) = 8\cos^4(t)$. Assinale na primeira coluna a frequência fundamental e na segunda e período fundamental de $f(t)$.

( ) $w = 1$

( ) $w = 2$

( ) $w = 4$

( ) $w = 2\pi$

( ) $w = \pi$

( ) $w = \frac{\pi}{2}$

( ) $T = 1$

( ) $T = 2$

( ) $T = 4$

( ) $T = 2\pi$

( ) $T = \pi$

( ) $T = \frac{\pi}{2}$

• **Questão 3** (1.0 ponto) Considere a função $f(t) = 8\cos^4(t)$. Calcule os coeficientes da expansão em série de Fourier de $f(t)$ e assinale na primeira coluna a representação trigonométrica e na segunda a representação exponencial.

( ) $3 + 8\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\cos(2nt) + \frac{n}{2n+1}\operatorname{sen}(2nt)\right)$

( ) $3 + \sum_{n=1}^{\infty}\frac{1}{2n}e^{2int}$

( ) $3 + 4\cos(2t) + \cos(4t)$

( ) $3 + 4\cos(t) + 2\cos(2t) + \cos(3t) + \frac{1}{2}\cos(4t)$

( ) $3 + 4\operatorname{sen}(t) + 2\operatorname{sen}(2t)$

( ) $\sum_{n=-\infty}^{\infty}\left(\frac{3}{2n+1} - \frac{in}{2n^2+1}\right)e^{2nit}$

( ) $3 + \sum_{n=1}^{\infty}\frac{1}{2n}\cos(2nt)$

( ) $\frac{i}{2}e^{-4it} + 2e^{-2it} + 3 + 2e^{2it} - \frac{i}{2}e^{4it}$

( ) $\frac{i}{2}e^{-2it} + 2ie^{-it} + 3 - 2ie^{it} - \frac{i}{2}e^{2it}$

( ) $\frac{1}{2}e^{-4it} + 2e^{-2it} + 3 + 2e^{2it} + \frac{1}{2}e^{4it}$

• **Questão 4** (1.0 ponto) Seja $f(t) = e^{-|t|}$ e $F(w) = \mathcal{F}\{f(t)\}$ e $g(t) := \mathcal{F}^{-1}\{iwF(w)e^{-iw}\}$. Assinale corretamente a alternativa que indica corretamente os valores de $g(2)$ e de $E := \frac{1}{\pi}\int_{-\infty}^{\infty}|F(w)|^2 dw$.

$g(2)$

( ) $-e^{-1}$

( ) e^{-1}

( ) e

( ) $-e$

( ) e^{-2}

( ) $-e^{-2}$

E

( ) 0

( ) 1

( ) 2

( ) 3

( ) 4

• **Questão 5** (1.0 pontos) Considere a função $f(t) = 1 + \text{sen}(2\pi t) + \cos(2\pi t) + \text{sen}(4\pi t)$. Assinale na primeira coluna o diagrama de espectro do módulo e, na segunda, o diagrama de espectro da fase.

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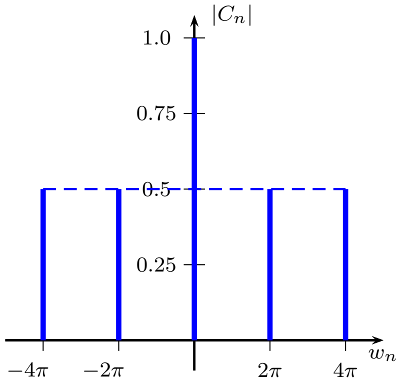

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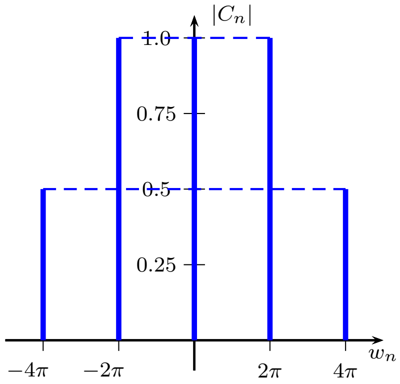

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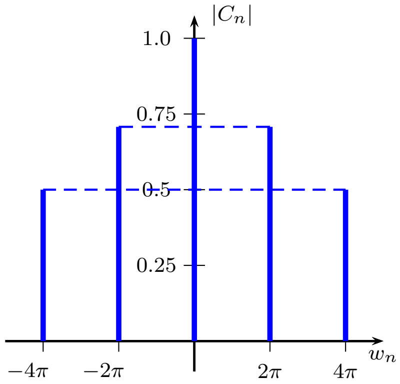

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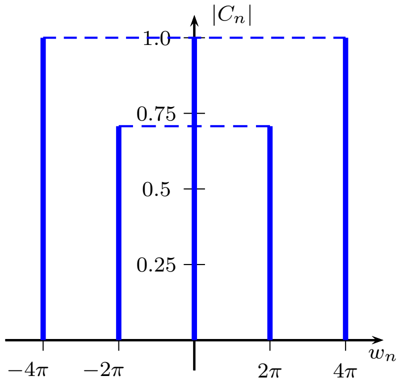

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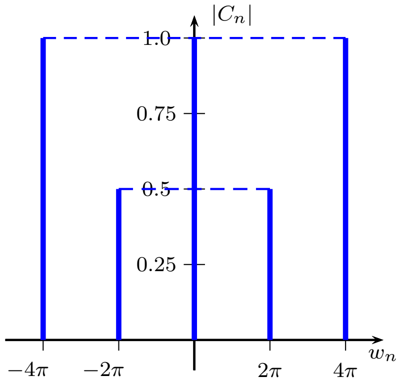

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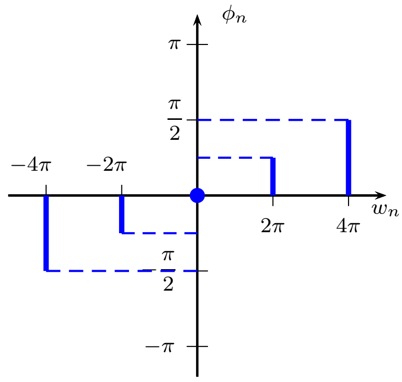

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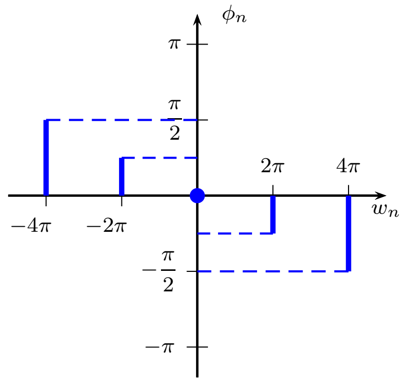

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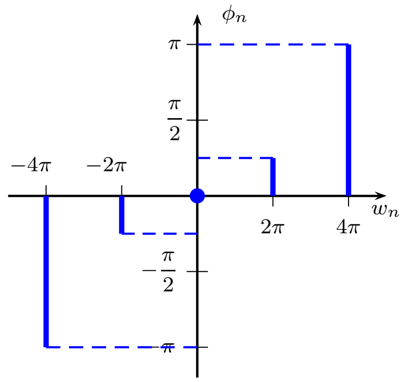

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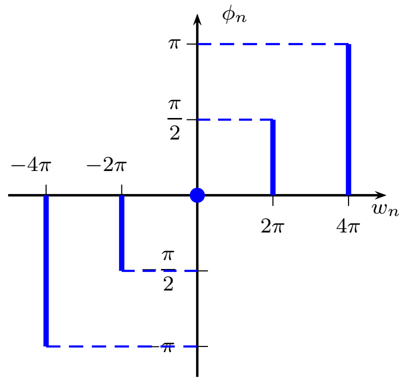

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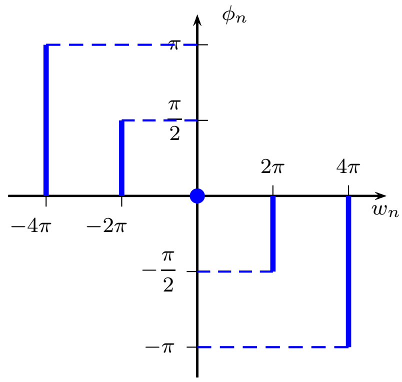
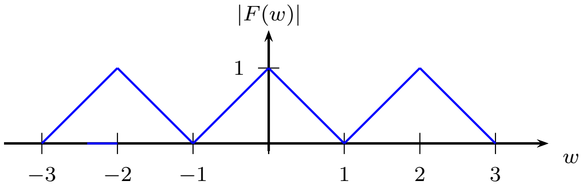


- **Questão 6** (2.5 ponto) Calcule a série de Fourier da função $f(t) = |\cos(\pi t)|$.

• **Questão 7** (2.5 pontos) Seja $f(t)$ uma função que possui transformada de Fourier $F(w) = \mathcal{F}\{f(t)\}$. O gráfico abaixo apresenta o diagrama de espectro de magnitudes de $F(w)$.



Esboce o diagrama de magnitudes de $g(t) = f'(t)\cos(3t)$ e $h(t) = \dfrac{d}{dt}\left(f(t)\cos(3t)\right)$.