UFRGS – INSTITUTO DE MATEMÁTICA
Departamento de Matemática Pura e Aplicada
MAT01168 - Turma A - 2016/1
Prova da área IA

1 - 6	7	8	Total

Nome: ______________________ **Cartão:** ____________

Regras Gerais:

- Não é permitido o uso de calculadoras, telefones ou qualquer outro recurso computacional ou de comunicação.
- Trabalhe individualmente e sem uso de material de consulta além do fornecido.
- Devolva o caderno de questões preenchido ao final da prova.

Regras para as questões abertas:

- Seja sucinto, completo e claro.
- Justifique todo procedimento usado.
- Indique identidades matemáticas usadas, em especial, itens da tabela.
- Use notação matemática consistente.

Identidades:

$\operatorname{sen}(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$	$\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$
$\operatorname{senh}(x) = \dfrac{e^{x} - e^{-x}}{2}$	$\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$
$(a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^{n-j} b^j, \quad \binom{n}{j} = \dfrac{n!}{j!(n-j)!}$	
$\operatorname{sen}(x+y) = \operatorname{sen}(x)\cos(y) + \operatorname{sen}(y)\cos(x)$	
$\cos(x+y) = \cos(x)\cos(y) - \operatorname{sen}(x)\operatorname{sen}(y)$	

Propriedades:

1	Linearidade	$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$
2	Transformada da derivada	$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$
3	Deslocamento no eixo s	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
4	Deslocamento no eixo t	$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$
5	Transformada da integral	$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{F(s)}{s}$
6	Filtragem da Delta de Dirac	$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$
7	Transformada da Delta de Dirac	$\mathcal{L}\{\delta(t-a)\} = e^{-as}$
8	Teorema da Convolução	$\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$, onde $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
9	Transformada de funções periódicas	$\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}}\int_0^T e^{-s\tau}f(\tau)d\tau$
10	Derivada da transformada	$\mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$
11	Integral da transformada	$\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \int_s^{\infty} F(s)ds$

Séries:

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots, \quad -1 < x < 1$
$\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots, \quad -1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots, \quad -\infty < x < \infty$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1}, \quad -1 < x < 1$
$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}, \quad -1 < x < 1$
$\operatorname{sen}(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
$\operatorname{senh}(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$
$\cosh(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$
$(1+x)^m = 1 + \sum_{n=1}^{\infty} \dfrac{m(m-1)\cdots(m-n+1)}{n!}x^n,$ $-1 < x < 1,\ m \neq 0, 1, 2, \ldots$

Funções especiais:

Função Gamma	$\Gamma(k) = \int_0^{\infty} x^{k-1}e^{-x}dx$
Propriedade da Função Gamma	$\Gamma(k+1) = k\Gamma(k), \quad k > 0$ $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$
Função de Bessel modificada de ordem ν	$I_\nu(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\dfrac{x}{2}\right)^{2m+\nu}$
Função de Bessel de ordem 0	$J_0(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{m!^2}\left(\dfrac{x}{2}\right)^{2m}$
Integral seno	$\operatorname{Si}(t) = \int_0^t \dfrac{\operatorname{sen}(x)}{x}dx$

Integrais:

$\int xe^{\lambda x}\,dx = \dfrac{e^{\lambda x}}{\lambda^2}(\lambda x - 1) + C$
$\int x^2 e^{\lambda x}\,dx = e^{\lambda x}\left(\dfrac{x^2}{\lambda} - \dfrac{2x}{\lambda^2} + \dfrac{2}{\lambda^3}\right) + C$
$\int x^n e^{\lambda x}\,dx = \dfrac{1}{\lambda}x^n e^{\lambda x} - \dfrac{n}{\lambda}\int x^{n-1}e^{\lambda x}dx + C$
$\int x\cos(\lambda x)\,dx = \dfrac{\cos(\lambda x) + \lambda x\operatorname{sen}(\lambda x)}{\lambda^2} + C$
$\int x\operatorname{sen}(\lambda x)\,dx = \dfrac{\operatorname{sen}(\lambda x) - \lambda x\cos(\lambda x)}{\lambda^2} + C$

Tabela de transformadas de Laplace:

	$F(s)=\mathcal{L}\{f(t)\}$	$f(t)=\mathcal{L}^{-1}\{F(s)\}$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^n}, \quad (n=1,2,3,...)$	$\frac{t^{n-1}}{(n-1)!}$
4	$\frac{1}{\sqrt{s}},$	$\frac{1}{\sqrt{\pi t}}$
5	$\frac{1}{s^{\frac{3}{2}}},$	$2\sqrt{\frac{t}{\pi}}$
6	$\frac{1}{s^k}, \quad (k>0)$	$\frac{t^{k-1}}{\Gamma(k)}$
7	$\frac{1}{s-a}$	e^{at}
8	$\frac{1}{(s-a)^2}$	te^{at}
9	$\frac{1}{(s-a)^n}, \quad (n=1,2,3...)$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
10	$\frac{1}{(s-a)^k}, \quad (k>0)$	$\frac{1}{\Gamma(k)}t^{k-1}e^{at}$
11	$\frac{1}{(s-a)(s-b)}, \quad (a\neq b)$	$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$
12	$\frac{s}{(s-a)(s-b)}, \quad (a\neq b)$	$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$
13	$\frac{1}{s^2+w^2}$	$\frac{1}{w}\operatorname{sen}(wt)$
14	$\frac{s}{s^2+w^2}$	$\cos(wt)$
15	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\operatorname{senh}(at)$
16	$\frac{s}{s^2-a^2}$	$\cosh(at)$
17	$\frac{1}{(s-a)^2+w^2}$	$\frac{1}{w}e^{at}\operatorname{sen}(wt)$
18	$\frac{s-a}{(s-a)^2+w^2}$	$e^{at}\cos(wt)$
19	$\frac{1}{s(s^2+w^2)}$	$\frac{1}{w^2}(1-\cos(wt))$
20	$\frac{1}{s^2(s^2+w^2)}$	$\frac{1}{w^3}(wt-\operatorname{sen}(wt))$
21	$\frac{1}{(s^2+w^2)^2}$	$\frac{1}{2w^3}(\operatorname{sen}(wt)-wt\cos(wt))$
22	$\frac{s}{(s^2+w^2)^2}$	$\frac{t}{2w}\operatorname{sen}(wt)$
23	$\frac{s^2}{(s^2+w^2)^2}$	$\frac{1}{2w}(\operatorname{sen}(wt)+wt\cos(wt))$
24	$\frac{s}{(s^2+a^2)(s^2+b^2)}, \quad (a^2\neq b^2)$	$\frac{1}{b^2-a^2}(\cos(at)-\cos(bt))$
25	$\frac{1}{(s^4+4a^4)}$	$\frac{1}{4a^3}[\operatorname{sen}(at)\cosh(at)- -\cos(at)\operatorname{senh}(at)]$
26	$\frac{s}{(s^4+4a^4)}$	$\frac{1}{2a^2}\operatorname{sen}(at)\operatorname{senh}(at))$
27	$\frac{1}{(s^4-a^2)}$	$\frac{1}{2a^3}(\operatorname{senh}(at)-\operatorname{sen}(at))$
28	$\frac{s}{(s^4-a^4)}$	$\frac{1}{2a^2}(\cosh(at)-\cos(at))$
29	$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$
30	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{\frac{-(a+b)t}{2}}I_0\left(\frac{a-b}{2}t\right)$
31	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
32	$\frac{s}{(s-a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
33	$\frac{1}{(s^2-a^2)^k}, \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}I_{k-\frac{1}{2}}(at)$
34	$\frac{1}{s}e^{-\frac{k}{s}}, \quad (k>0)$	$J_0(2\sqrt{kt})$
35	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos(2\sqrt{kt})$
36	$\frac{1}{s^{\frac{3}{2}}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\operatorname{senh}(2\sqrt{kt})$
37	$e^{-k\sqrt{s}}, \quad (k>0)$	$\frac{k}{2\sqrt{\pi t^3}}e^{-\frac{k^2}{4t}}$
38	$\frac{1}{s}\ln(s)$	$-\ln(t)-\gamma, \quad (\gamma\approx 0,5772)$
39	$\ln\left(\frac{s-a}{s-b}\right)$	$\frac{1}{t}\left(e^{bt}-e^{at}\right)$
40	$\ln\left(\frac{s^2+w^2}{s^2}\right)$	$\frac{2}{t}(1-\cos(wt))$
41	$\ln\left(\frac{s^2-a^2}{s^2}\right)$	$\frac{2}{t}(1-\cosh(at))$
42	$\tan^{-1}\left(\frac{w}{s}\right)$	$\frac{1}{t}\operatorname{sen}(wt)$
43	$\frac{1}{s}\cot^{-1}(s)$	$\operatorname{Si}(t)$
44	$\frac{1}{s}\tanh\left(\frac{as}{2}\right)$	Onda quadrada $f(t)=\begin{cases}1, & 0<t<a\\ -1, & a<t<2a\end{cases}$ $f(t+2a)=f(t), \quad t>0$
45	$\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$	Onda triangular $f(t)=\begin{cases}\frac{t}{a}, & 0<t<a\\ -\frac{t}{a}+2, & a<t<2a\end{cases}$ $f(t+2a)=f(t), \quad t>0$
46	$\frac{w}{(s^2+w^2)\left(1-e^{-\frac{\pi}{w}s}\right)}$	Retificador de meia onda $f(t)=\begin{cases}\operatorname{sen}(wt), & 0<t<\frac{\pi}{w}\\ 0, & \frac{\pi}{w}<t<\frac{2\pi}{w}\end{cases}$ $f\left(t+\frac{2\pi}{w}\right)=f(t), \quad t>0$
47	$\frac{w}{s^2+w^2}\coth\left(\frac{\pi s}{2w}\right)$	Retificador de onda completa $f(t)=\lvert\operatorname{sen}(wt)\rvert$
48	$\frac{1}{as^2}-\frac{e^{-as}}{s(1-e^{-as})}$	Onda dente de serra $f(t)=\frac{t}{a}, \quad 0<t<a$ $f(t)=f(t-a), \quad t>a$

- **Questão 1** (1.0 ponto) A transformada de Laplace da função $t^2u(t-2)$ é

( ) $\dfrac{2}{s^3}$

( ) $\dfrac{2}{s^3}e^{-2s}$

( ) $\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}\right)e^{-2s}$

(X) $\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right)e^{-2s}$

( ) $\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}+1\right)e^{-2s}$

- **Questão 2** (1.0 ponto) A transformada de Laplace da função $\dfrac{\operatorname{senh}(t)}{t}$ é

( ) $\dfrac{1}{2}\ln\left(\dfrac{s-1}{s+1}\right)$

(X) $\dfrac{1}{2}\ln\left(\dfrac{s+1}{s-1}\right)$

( ) $\ln\left(\dfrac{s+1}{s-1}\right)$

( ) $\dfrac{1}{s^2-1}e^{-s}$

( ) $\dfrac{1}{s}\dfrac{1}{s^2-1}$

- **Questão 3** (1.0 ponto) Sabendo que $\mathcal{L}\{f(t)\} = F(s)$ é correto afirmar que

( ) $\dfrac{d^2F(s)}{ds^2} = s^2\mathcal{L}\{f(t)\}$

( ) $\dfrac{d^2F(s)}{ds^2} = -s^2\mathcal{L}\{f(t)\}$

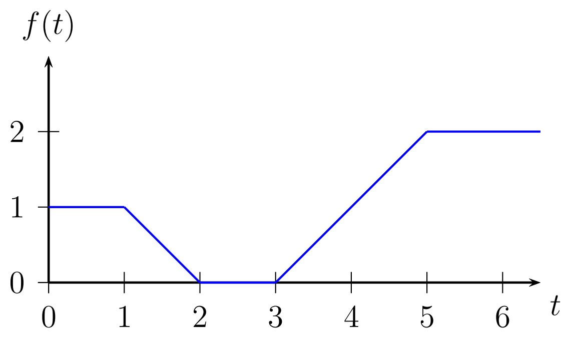
( ) $\dfrac{d^2F(s)}{ds^2} = -\mathcal{L}\{tf(t)\}$

( ) $\dfrac{d^2F(s)}{ds^2} = -\mathcal{L}\{t^2f(t)\}$

(X) $\dfrac{d^2F(s)}{ds^2} = \mathcal{L}\{t^2f(t)\}$

( ) $\dfrac{d^2F(s)}{ds^2} = \mathcal{L}\{f''(t)\}$

- **Questão 4** (1.0 ponto) Considere a função $f : \mathbb{R}_+ \to \mathbb{R}$ dada no gráfico abaixo:



A transformada de Laplace da função $f(t)$ é

( ) $\dfrac{-e^{-s}+e^{-2s}+e^{-3s}-e^{-5s}}{s^2}$

( ) $\dfrac{-e^{-s}+e^{-2s}+e^{-3s}-e^{-5s}}{s}$

(X) $\dfrac{s-e^{-s}+e^{-2s}+e^{-3s}-e^{-5s}}{s^2}$

( ) $\dfrac{1-e^{-s}+e^{-2s}+e^{-3s}-e^{-5s}}{s^2}$

( ) $\dfrac{1-e^{-s}+e^{-2s}+e^{-3s}-e^{-5s}}{s}$

• **Questão 5** (1.0 pontos) Dado que $f(t)$ satisfaz a equação

$$f(t)+e^t\int_0^t e^{-\tau}f(\tau)d\tau=\mathrm{senh}(t)$$

então a transformada de Laplace de f é

(X) $F(s)=\dfrac{1}{s(s+1)}$

( ) $F(s)=\dfrac{1}{(s-1)^2+1}$

( ) $F(s)=\dfrac{1}{s(s-1)}$

( ) $F(s)=\dfrac{1}{s^2-1}$

( ) $F(s)=\dfrac{1}{s^2+1}$

( ) $F(s)=\dfrac{1}{s-1}$

• **Questão 6** (1.0 ponto) Considere a função $F(s)=e^{-sk}\dfrac{a+bs}{s^2+cs+d}$ para constantes a e b reais, $c\geq 0$, $k\geq 0$ e $d>0$. Marque qual gráfico abaixo certamente **NÃO** pode representar a transformada inversa da função $F(s)$.

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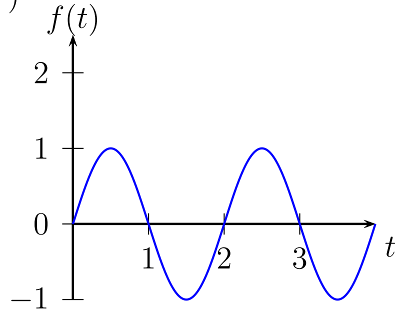

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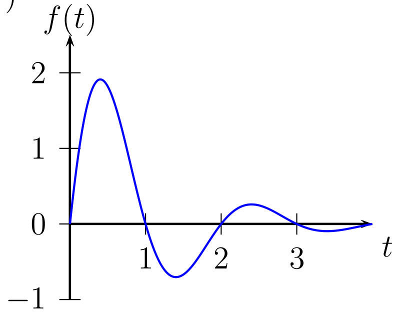

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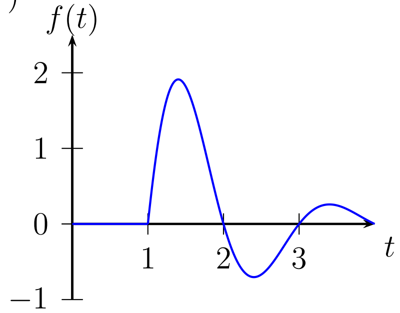

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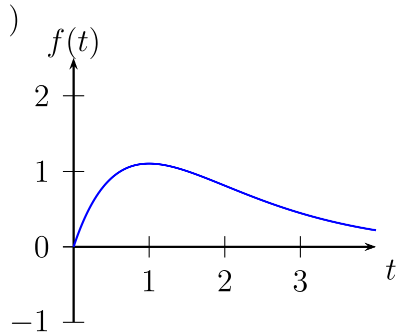

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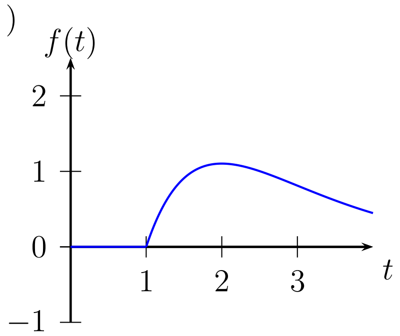
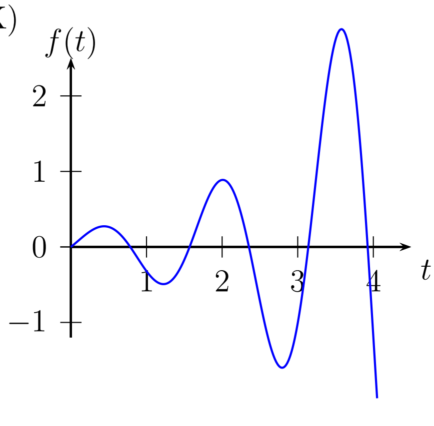


(X)

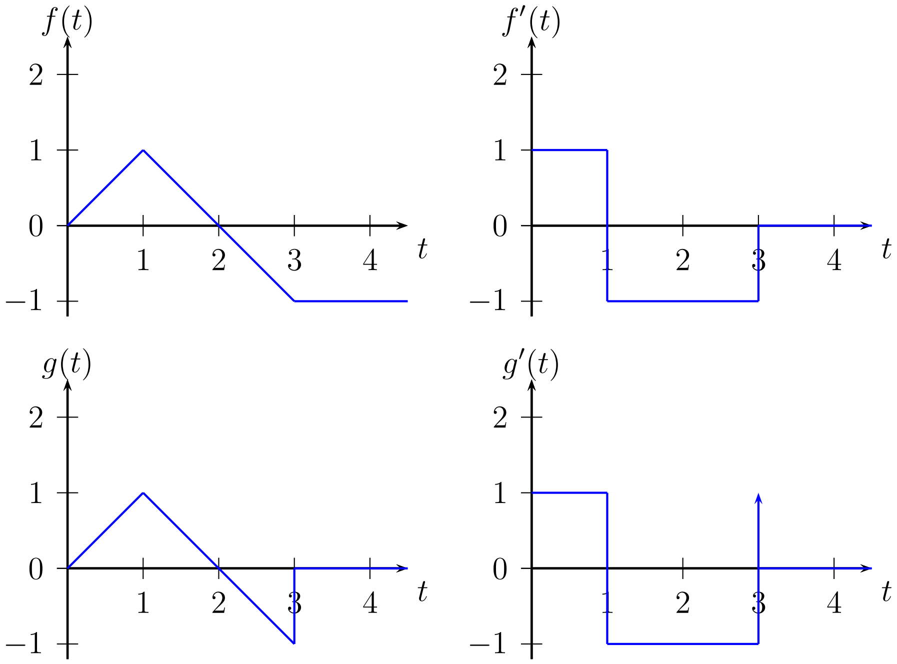


• **Questão 7** (2.0 ponto) Considere as funções $f(t) = tu(t) + (2-2t)u(t-1) + (t-3)u(t-3)$ e $g(t) = tu(t) + (2-2t)u(t-1) + (t-2)u(t-3)$

a) (1.0 pontos) Esboce os gráficos de f, g, f' e g'.

b) (1.0 pontos) Calcule $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{f'(t)\}$, $\mathcal{L}\{g(t)\}$ e $\mathcal{L}\{g'(t)\}$.

Solução: a)



b)

$$\mathcal{L}\{f'(t)\} = \frac{1-2e^{-s}+e^{-3s}}{s}$$
$$\mathcal{L}\{f(t)\} = \frac{1-2e^{-s}+e^{-3s}}{s^2}$$
$$\mathcal{L}\{g'(t)\} = \frac{1-2e^{-s}+e^{-3s}+se^{-3s}}{s}$$
$$\mathcal{L}\{g(t)\} = \frac{1-2e^{-s}+e^{-3s}+se^{-3s}}{s^2}$$

• **Questão 8** (2.0 ponto) Considere o oscilador harmônico

$$\begin{cases} y'' + 4y = \operatorname{sen}(w_0 t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

onde w_0 é uma constante positiva.

a) (1.0 pontos) Resolva o problema de valor inicial para $w_0 = 2$.

b) (1.0 pontos) Resolva o problema de valor inicial para $w_0 \neq 2$.

Solução: a) Usamos a transformada de Laplace para resolver o PVI:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{2}{s^2+4} \\ &\Downarrow \\ Y(s) &= \frac{2}{(s^2+4)(s^2+4)} \\ &\Downarrow \\ Y(s) &= \frac{2}{(s^2+4)^2} \end{aligned}$$

Pelo item 21 da tabela, temos:

$$y(t) = \frac{1}{8}\left(\operatorname{sen}(2t) - 2t\cos(2t)\right)$$

b) Usamos a transformada de Laplace para resolver o PVI com $w_0 \neq 2$:

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{w_0}{s^2 + w_0^2} \\
&\Downarrow \\
Y(s) &= \frac{w_0}{(s^2+4)(s^2+w_0^2)} \\
&\Downarrow \\
Y(s) &= \frac{w_0}{2(w_0^2-4)}\frac{2}{s^2+4} + \frac{1}{4-w_0^2}\frac{w_0}{s^2+w_0^2}
\end{aligned}$$

Pelo item 21 da tabela, temos:

$$y(t) = \frac{w_0}{2(w_0^2-4)}\operatorname{sen}(2t) + \frac{1}{4-w_0^2}\operatorname{sen}(w_0 t)$$